South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 3 (2024), pp. 103-122

DOI: 10.56827/SEAJMMS.2024.2003.8 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

INEQUALITIES FOR HIGHER ORDER ITERATED DIFFERENCE EQUATIONS THROUGH SYMMETRIC FUNCTIONS

Nagesh Kale

Department of Mathematics,
Sant Rawool Maharaj Mahavidyalaya Kudal,
Sindhudurg - 416520, Maharashtra, INDIA

E-mail : nageshkale7991@gmail.com

(**Received:** *Nov. 09, 2023* **Accepted:** *Dec. 13, 2024* **Published:** *Dec. 30, 2024)*

Abstract: The present research offers insight into the connections between higher-order difference equations with iterated sums and symmetric functions, as well as the resulting inequalities from these interactions. In this study, we introduce new difference inequalities of an algebraic nature that use symmetric functions to handle significant higher-order nonlinear finite difference equations with iterated sums. With the aid of these results, it will be simple to examine a collection of higher-order nonlinear finite difference equations with iterated sums. Finally, we study the boundedness, uniqueness, and continuous dependency of the solution on the initial data of a class of iterated difference equations. Furthermore, we present a numerical illustration to highlight the relevance of our findings.

Keywords and Phrases: Symmetric functions, Difference inequalities, Higher-order difference equations, Iterated sums.

2020 Mathematics Subject Classification: 26D15, 26D07, 39A22, 39A13, 39B12.

1. Introduction

Difference equations and symmetric functions are fundamental mathematical notions with distinct properties and applications. Difference equations reflect the evolution of discrete sequences, whereas symmetric functions represent the symmetry of algebraic expressions. This article investigates the relationships between these seemingly disparate mathematical notions, which lead to numerous inequalities. Our research intends to systematically analyze and implement these inequalities by combining difference equations with iterated sums and symmetric functions.

Solution analysis for differential, integral, and difference equations relies heavily on inequalities for a wide range of qualitative and quantitative purposes. The following Gronwall-Bellman inequality [7] is one of the most commonly used inequalities in the research of differential equations.

If $u(s)$ and $p(s)$ are continuous non-negative functions on $[\alpha, \beta]$ such that

$$u(s) \leq u_0 + \int_{\alpha}^{s} p(t)u(t)dt, \text{ for } s \in [\alpha, \beta],$$

where $u_0 \in [0, \infty)$, then

$$u(s) \leq u_0 \exp\left(\int_{\alpha}^{s} p(t)dt\right), \text{ for } s \in [\alpha, \beta].$$

In the later decades, several generalizations and extensions of this inequality have appeared in literature to explore more general classes of integral and differential equations (see [10, 14, 18, 21] and references therein). In [13], L. Ou-Yang developed a crucial generalization of Gronwall-Bellman inequality to its nonlinear form stated below.

If $u(s)$ and $p(s)$ are non-negative functions defined on $[0, \infty)$ such that

$$u^2(s) \leq u_0^2 + \int_{0}^{s} p(t)u(t)dt, \text{ for } s \in [0, \infty),$$

where $u_0 \in [0, \infty)$, then

$$u(s) \leq u_0 + \int_{0}^{s} p(t)dt, \text{ for } s \in [0, \infty).$$

Recently, Abdeldaim and Yakout [1] developed a more extended version of Gronwall-Bellman inequality to uncover the hidden characteristics of solutions of several families of integral equations.

Simultaneously, several researchers devised numerous discrete analogues of the above-mentioned and similar classes of inequalities to undertake the research of finite difference equations and their solutions (see [3, 4, 8, 9, 10, 11, 17, 18, 20, 22] and references mentioned therein).

Though a handful of inequalities related to above-mentioned classes have been investigated, extended, and generalized, these inequalities were not adequate to deal with classes of higher-order difference equations. In [12], Medved generalized the Bihari-like integral inequalities with double integrals due to various researchers [2, 5, 6, 19] to multiple integrals with delay to encompass the study of higher order differential equations and integrodifferential equations. Along similar lines, Pachpatte [14, 16] initiated the study of a class higher order integrodifferential equations in one variable with iterated integrals. To analyze several aspects of the solutions of these equations, Pachpatte developed several integral inequalities and studied the boundedness and uniqueness of the solutions. In the later years, Pachpatte also presented the discrete analogues of these inequalities (see [15, 17]) to furnish a potent tool to deal with higher-order linear difference equations.

These results, however, do not obtain clear estimates of the solutions of some more general families of finite difference equations with iterated sums. In this manuscript, we present a few new inequalities of an algebraic type involving symmetric functions to address the more significant and general class of the aforementioned difference equations. We also explore solutions of a class of higher-order difference equations with iterated sums for uniqueness, boundedness, and its continuous dependence on the original data. We also present a numerical illustration to highlight the usefulness of our result.

In the subsequent discussion, we set $N_0 = \{0, 1, 2, \dots\}$ and define an operator L on class of non-negative functions $u(s)$ on N_0 (see [17]) recursively as

$$\begin{aligned} L_0 u(s) &= u(s), \\ L_t u(s) &= \frac{L_{t-1}u(s+1) - L_{t-1}u(s)}{w_t(s)} \text{ for } t \geq 1, \end{aligned}$$

where $u(s)$ is non-negative and $w_t(s)$ are some positive functions defined on N_0 for $1 \leq t \leq k$ with $w_k(s) = 1$. We list here an important Pachpatte's inequality [17], which is crucial for our proofs.
If $u(s), a(s), b(s)$ are non-negative functions on N_0 with

$$\Delta u(s) \leq a(s)u(s) + b(s),$$

then

$$u(s) \leq u(0) \prod_{s=0}^{t-1}(1 + a(s)) + \sum_{s=0}^{t-1} b(s) \prod_{s'=s+1}^{t-1} (1 + a(s')).$$

2. Main results

In the subsequent lemma, we develop symmetric function method and obtain a set of connections between an unknown function and its earlier values.

Lemma 2.1. *If*

$$u(m+1) \leq (\alpha\ v(m)+1)u(m)+\beta\ v(m), \quad u(0)=u_0, \tag{2.1}$$

for some nonnegative functions $u(m), v(m)$ defined on N_0, then

$$u(m) \leq u_0 \prod_{n=0}^{m-1}(1+\alpha v(n))+\beta \sum_{n=0}^{m-1} \alpha^n S_{n+1}, \tag{2.2}$$

where $S_1, S_2, \dots, S_m$ are elementary symmetric functions defined as

$$\begin{aligned} S_1 &= v(0)+v(1)+\cdots+v(m-1), \\ S_2 &= v(0)v(1)+v(0)v(2)+\cdots+v(m-2)v(m-1), \\ &\vdots \\ S_m &= v(0)v(1)\dots v(m-1). \end{aligned}$$

Proof. We have $m \geq 0$. Now, we will use the idea of recurrence to accomplish this result.
Firstly, for $m=0$, inequality (2.1) gives the estimate

$$u(1) \leq (\alpha\ v(0)+1)u(0)+\beta\ v(0). \tag{2.3}$$

If we put $m=1$ in the inequality (2.1), we get

$$u(2) \leq (\alpha\ v(1)+1)u(1)+\beta\ v(1). \tag{2.4}$$

Now, using (2.3) in (2.4), we obtain

$$\begin{aligned} u(2) &\leq (\alpha v(1)+1)((\alpha v(0)+1)u(0)+\beta v(0))+\beta v(1) \\ &= (\alpha v(1)+1)(\alpha v(0)+1)u(0)+(\alpha v(1)+1)\beta v(0)+\beta v(1) \\ &= (\alpha v(1)+1)(\alpha v(0)+1)u(0)+\beta(\alpha v(0)v(1)+(v(0)+v(1))). \end{aligned} \tag{2.5}$$

Proceeding identically, we obtain

$$\begin{aligned} u(m) &\leq (\alpha v(0)+1)(\alpha v(1)+1)\dots(\alpha v(m-1)+1)u(0) \\ &\quad +\beta(\alpha^{m-1}v(0)v(1)\dots v(m-1)+\cdots+\alpha^0(v(0)+v(1)+\cdots+v(m-1))). \end{aligned} \tag{2.6}$$

Consider elementary symmetric polynomials for $v(0), v(1), \ldots, v(m-1)$ as in the statement. Then, we find that

$$u(m) \leq u_0 \prod_{n=0}^{m-1}(1+\alpha v(n)) + \beta \sum_{n=0}^{m-1} \alpha^n S_{n+1}.$$

This completes the proof of our lemma.

Remark 2.1. *In nonlinear analysis, geometric convergence rates are frequently examined about iterative PDE solvers, such as the Poisson-Nernst-Planck equations and preconditioned steepest descent for p-Laplacian terms. These techniques aim for minimal computational steps and effective convergence to a solution. However, the iterative structure developed in this paper, which involves symmetric functions and iterated sums, differs in approach and context from the previously described nonlinear iterative methods.*

In our next results, we use the symmetric function approach outlined in the prior lemma and the subsequent inequality to develop power non-linear estimates for higher-order difference equations involving iterated sums.

Lemma 2.2. (Zhao [21]) *If* $x \geq 0, p_1 \geq p_2 \geq 0$, *where* $p_1 \neq 0$, *then* $x^{\frac{p_2}{p_1}} \leq \frac{p_2}{p_1} c^{\frac{p_2-p_1}{p_1}} x + \frac{p_1-p_2}{p_1} c^{\frac{p_2}{p_1}}$, *for any* $c > 0$.

In the further results, we develop $p_1 - p_2-$nonlinear variants of higher order difference equations involving iterated sums.

Theorem 2.3. *Assume that* $u(m), f(m)$ *are nonnegative functions and* $w_1(m)$, $w_2(m), \ldots, w_{k-1}(m)$ *are positive functions for* $m \in N_0$. *If* $p_1 \geq p_2 \geq 0, p_1 \neq 0$ *and for some nonnegative constant* u_0,

$$u^{p_1}(m) \leq u_0 + \sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m_k) u^{p_2}(m_k), \ m \in N_0, \tag{2.7}$$

then

$$u(m) \leq \left\{ u_0 \prod_{n=0}^{m-1}(1+\lambda_1 \tilde{f}(n)) + \lambda_2 \sum_{n=0}^{m-1} \lambda_1^n S_{1,n+1} \right\}^{\frac{1}{p_1}}, \tag{2.8}$$

where

$$\left.\begin{aligned}
\lambda_1 &= \frac{p_2}{p_1}c^{\frac{p_2-p_1}{p_1}}, \lambda_2 = \frac{p_1-p_2}{p_1}c^{\frac{p_2}{p_1}},\ c>0,\\
S_{1,1} &= \tilde{f}(0)+\tilde{f}(1)+\cdots+\tilde{f}(m-1),\\
S_{1,2} &= \tilde{f}(0)\tilde{f}(1)+\tilde{f}(0)\tilde{f}(2)+\cdots+\tilde{f}(m-2)\tilde{f}(m-1),\\
&\vdots\\
S_{1,m} &= \tilde{f}(0)\tilde{f}(1)\ldots\tilde{f}(m-1),\ \text{and}\\
\tilde{f}(m) &= w_1(m)\sum_{m_2=0}^{m-1}w_2(m_2)\sum_{m_3=0}^{m_2-1}w_3(m_3)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1}w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1}f(m_k).
\end{aligned}\right\} \tag{2.9}$$

Theorem 2.4. *Assume that $u(m), f(m)$ are nonnegative functions and $g(m)$ is a positive function for $m \in N_0$. If $p_1 \geq p_2 \geq 0, p_1 \neq 0$ and for some nonnegative constant u_0,*

$$u^{p_1}(m) \leq u_0 + \sum_{m_1=0}^{m-1}\sum_{m_2=0}^{m_1-1}\sum_{m_3=0}^{m_2-1}\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1}\frac{1}{g(m_{k-1})}\sum_{m_k=0}^{m_{k-1}-1}f(m_k)u^{p_2}(m_k),\ m\in N_0, \tag{2.10}$$

then

$$u(m) \leq \left\{u_0\prod_{n=0}^{m-1}(1+\lambda_1\phi(n))+\lambda_2\sum_{n=0}^{m-1}\lambda_1^n S_{2,n+1}\right\}^{\frac{1}{p_1}}, \tag{2.11}$$

where λ_1, λ_2 are as defined in Theorem 2.3,

$$\left.\begin{aligned}
S_{2,1} &= \phi(0)+\phi(1)+\cdots+\phi(m-1),\\
S_{2,2} &= \phi(0)\phi(1)+\phi(0)\phi(2)+\cdots+\phi(m-2)\phi(m-1),\\
&\vdots\\
S_{2,m} &= \phi(0)\phi(1)\ldots\phi(m-1),\quad \text{and}\\
\phi(m) &= \sum_{m_2=0}^{m-1}\sum_{m_3=0}^{m_2-1}\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1}\frac{1}{g(m_{k-1})}\sum_{m_k=0}^{m_{k-1}-1}f(m_k).
\end{aligned}\right\} \tag{2.12}$$

Theorem 2.5. *Assume that $u(m), f(m)$ are nonnegative functions and $g(m)$ is a positive function defined on N_0. If $p_1 \geq p_2 \geq 0, p_1 \neq 0$ and for some nonnegative constant u_0,*

$$u^{p_1}(m) \leq u_0 + \sum_{m_1=0}^{m-1} \frac{1}{g(m_1)} \sum_{m_2=0}^{m_1-1} \sum_{m_3=0}^{m_2-1} \cdots \sum_{m_k=0}^{m_{k-1}-1} f(m_k) u^{p_2}(m_k), \;\; m \in N_0, \tag{2.13}$$

then

$$u(m) \leq \left\{ u_0 \prod_{n=0}^{m-1} (1 + \lambda_1 \psi(n)) + \lambda_2 \sum_{n=0}^{m-1} \lambda_1^n S_{3,n+1} \right\}^{\frac{1}{p_1}}, \tag{2.14}$$

where λ_1, λ_2 *are as defined in Theorem 2.3,*

$$\left.\begin{aligned} S_{3,1} &= \psi(0) + \psi(1) + \cdots + \psi(m-1), \\ S_{3,2} &= \psi(0)\psi(1) + \psi(0)\psi(2) + \cdots + \psi(m-2)\psi(m-1), \\ &\vdots \\ S_{3,m} &= \psi(0)\psi(1) \ldots \psi(m-1), \quad \text{and} \\ \psi(m) &= \frac{1}{g(m)} \sum_{m_2=0}^{m-1} \sum_{m_3=0}^{m_2-1} \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} \sum_{m_k=0}^{m_{k-1}-1} f(m_k). \end{aligned}\right\} \tag{2.15}$$

Theorem 2.6. *Assume that* $u(m), f(m)$ *are nonnegative functions and* $g(m)$ *is a positive function defined on* N_0*. If* $p_1 \geq p_2 \geq 0, p_1 \neq 0$ *and for some nonnegative constant* u_0*,*

$$u^{p_1}(m) \leq u_0 + \sum_{m_1=0}^{m-1} \sum_{m_2=0}^{m_1-1} \sum_{m_3=0}^{m_2-1} \cdots \sum_{m_k=0}^{m_{k-1}-1} \frac{1}{g(m_k)} \sum_{n_1=0}^{m_k-1} \sum_{n_2=0}^{n_1-1} \sum_{n_3=0}^{n_2-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} f(n_k) u^{p_2}(n_k), \;\; m \in N_0, \tag{2.16}$$

then

$$u(m) \leq \left\{ u_0 \prod_{n=0}^{m-1} (1 + \lambda_1 \chi(n)) + \lambda_2 \sum_{n=0}^{m-1} \lambda_1^n S_{4,n+1} \right\}^{\frac{1}{p_1}}, \tag{2.17}$$

where λ_1, λ_2 *are as defined in Theorem 2.3,*

$$\left.\begin{aligned} S_{4,1} &= \chi(0) + \chi(1) + \cdots + \chi(m-1), \\ S_{4,2} &= \chi(0)\chi(1) + \chi(0)\chi(2) + \cdots + \chi(m-2)\chi(m-1), \\ &\vdots \\ S_{4,m} &= \chi(0)\chi(1)\ldots\chi(m-1), \text{ and} \\ \chi(m) &= \sum_{m_2=0}^{m_1-1}\sum_{m_3=0}^{m_2-1}\cdots\sum_{m_k=0}^{m_{k-1}-1}\frac{1}{g(m_k)}\sum_{n_1=0}^{m_k-1}\sum_{n_2=0}^{n_1-1}\sum_{n_3=0}^{n_2-1}\cdots\sum_{n_k=0}^{n_{k-1}-1} f(n_k). \end{aligned}\right\} \tag{2.18}$$

Proof. The proofs for Theorems 2.3 to 2.6 exhibit a comparable structure; so, we shall provide the proof for Theorem 2.3 as an exemplification. Analogous procedures can be employed, with suitable revisions, to establish the proofs for the other theorems.

Let's start with the condition in which $u_0 > 0$ and set

$$z(m) = u_0 + \sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m_k) u^{p_2}(m_k). \tag{2.19}$$

It provides

$$z(0) = u_0, \tag{2.20}$$

and the inequality (2.7) implies that

$$u^{p_1}(m) \le z(m). \tag{2.21}$$

Further, using Zhao's Lemma 2.2, we obtain

$$L_k z(m) = f(m)u^{p_2}(m) \le f(m) z^{\frac{p_2}{p_1}}(m) \le f(m)(\lambda_1 z(m) + \lambda_2). \tag{2.22}$$

Obviously, $z(m) \le z(m+1)$ and hence, we get

$$\begin{aligned} \Delta\left(\frac{L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2}\right) &= \frac{L_{k-1}z(m+1)}{\lambda_1 z(m+1) + \lambda_2} - \frac{L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2} \\ &\le \frac{L_{k-1}z(m+1)}{\lambda_1 z(m) + \lambda_2} - \frac{L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2} \\ &= \frac{\Delta L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2} \\ &= \frac{L_k z(m)}{\lambda_1 z(m) + \lambda_2} \le f(m). \end{aligned} \tag{2.23}$$

Now, substituting $m = m_k$ and summing both sides of inequality (2.23) over m_k from 0 to $m-1$ leads to

$$\frac{L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2} \leq \sum_{m_k=0}^{m-1} f(m_k). \tag{2.24}$$

Further,

$$\Delta\left(\frac{L_{k-2}z(m)}{\lambda_1 z(m) + \lambda_2}\right) \leq \frac{\Delta L_{k-2}z(m)}{\lambda_1 z(m) + \lambda_2} = \frac{w_{k-1}(m)L_{k-1}z(m)}{\lambda_1 z(m) + \lambda_2} \leq w_{k-1}(m)\sum_{m_k=0}^{m-1} f(m_k). \tag{2.25}$$

At this time, put $m = m_{k-1}$ in the inequality (2.25) and sum it over m_{k-1} from 0 to $m-1$ to get

$$\frac{L_{k-2}z(m)}{\lambda_1 z(m) + \lambda_2} \leq \sum_{m_{k-1}=0}^{m-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m_k). \tag{2.26}$$

Proceeding further in the same fashion, we obtain

$$\frac{\Delta z(m)}{\lambda_1 z(m) + \lambda_2} \leq w_1(m)\sum_{m_2=0}^{m-1} w_2(m_2)\sum_{m_3=0}^{\lambda_2-1} w_3(m_3)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1} f(m_k). \tag{2.27}$$

It provide us the estimate

$$\begin{aligned} z(m+1) \leq & \left\{ w_1(m)\sum_{m_2=0}^{m-1} w_2(m_2)\sum_{m_3=0}^{\lambda_2-1} w_3(m_3)\dot{\sum}_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1} f(m_k)\right\} \\ & \times (\lambda_1 z(m) + \lambda_2) + z(m) \\ & = \left(\lambda_1 \tilde{f}(m) + 1\right) z(m) + \lambda_2 \tilde{f}(m), \end{aligned} \tag{2.28}$$

where $\tilde{f}(m)$ is as stated in (2.9). Thus, using Lemma 2.1, we achieve the bound on $z(m)$ as

$$z(m) \leq z(0)\prod_{n=0}^{m-1}(1 + \lambda_1 \tilde{f}(n)) + \lambda_2 \sum_{n=0}^{m-1} \lambda_1^n S_{1,n+1}, \tag{2.29}$$

where $S_{1,1}, S_{1,2}, \dots, S_{1,m}$ are elementary symmetric functions as defined in (2.9). Finally, the combined use of (2.21) and (2.29) provides the desired explicit bound on $u(m)$ as

$$u(m) \le \left\{ u_0 \prod_{n=0}^{m-1} (1 + \lambda_1 \tilde{f}(n)) + \lambda_2 \sum_{n=0}^{m-1} \lambda_1^n S_{1,n+1} \right\}^{\frac{1}{p_1}}.$$

This concludes the proof of our inequality.

Remark 2.2. *In the next remarks, we present many observations that underscore the importance of our findings in relation to the existing literature.*

(i) Pachpatte's inequality ([17], p. 46, Theorem 1.6.1(a_1)) becomes a particular case of Theorem 2.3 for $p_1 = p_2 = 1$.

(ii) The inequality presented in Theorem 2.4 extends the Pachpatte's inequality stated in ([17], p. 46, Theorem 1.6.1(a_2)) to more general nonlinear case. Moreover, considering $p_1 = p_2 = 1$, one can conveniently achieve this noted Pachpatte's inequality.

(iii) If one assumes $p_1 = 1 = p_2$, then the inequality in Theorem 2.5 turns into a more general nonlinear extension of Pachpatte's inequality ([17], p. 47, Theorem 1.6.1(a_3)).

(iv) The inequality in Theorem 2.6 becomes a more general nonlinear extension of Pachpatte's inequality ([17], p. 47, Theorem 1.6.1(a_4)) if $p_1 = p_2 = 1$ is assumed.

Theorem 2.7. *Assume that $u(m), f_1(m), f_2(m), \dots, f_t(m)$ are non-negative functions and $w_1(m), w_2(m), \dots, w_t(m)$ are positive functions on N_0. If $p_1 \ge p_2 \ge 0, p_1 \ne 0$ and for some $u_0 \ge 0$,*

$$u^{p_1}(m) \le u_0 + \sum_{k=1}^{t} \left(\sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} w_k(m_k) f_k(m_k) u^{p_2}(m_k) \right), \text{ with } m_0 = m,\ m \in N_0, \tag{2.30}$$

then

$$u(m) \le \left\{ u_0 \prod_{m_1=0}^{m-1} (1 + \lambda_1 F(m_1)) + \sum_{m_1=0}^{m-1} \lambda_2 F(m_1) \prod_{n=m_1+1}^{m-1} (1 + \lambda_1 F(n)) \right\}^{\frac{1}{p_1}},$$

where λ_1, λ_2 are as mentioned in Theorem 2.3 and

$$F(m) = w_1(m)f_1(m) + w_1(m) \times \sum_{k=2}^{t}\left(\sum_{m_2=0}^{m-1} w_2(m_2)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1} w_k(m_k)f_k(m_k)\right).$$

Proof. Let us begin with $u_0 > 0$, and let $z(m)$ denote right hand side of (2.30). Then, we observe that

$$\frac{\Delta z(m)}{w_1(m)} - f_1(m)u^{p_2}(m) = \sum_{k=2}^{t}\left(\sum_{m_2=0}^{m_1-1} w_2(m_2)\cdots\sum_{m_k=0}^{m_{k-1}-1} w_k(m_k)f_k(m_k)u^{p_2}(m_k)\right). \quad (2.31)$$

Set

$$z_1(m) = \sum_{k=2}^{t}\left(\sum_{m_2=0}^{m_1-1} w_2(m_2)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1} w_k(m_k)f_k(m_k)u^{p_2}(m_k)\right). \quad (2.32)$$

On the previous lines, we find that

$$\frac{\Delta z_1(m)}{w_2(m)} - f_2(m)u^{p_2}(m) = \sum_{k=3}^{t}\left(\sum_{m_3=0}^{m_2-1} w_3(m_3)\cdots\sum_{m_k=0}^{m_{k-1}-1} w_k(m_k)f_k(m_k)u^{p_2}(m_k)\right). \quad (2.33)$$

Consider

$$z_2(m) = \sum_{k=3}^{t}\left(\sum_{m_3=0}^{m_2-1} w_3(m_3)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1} w_k(m_k)f_k(m_k)u^{p_2}(m_k)\right). \quad (2.34)$$

Proceeding similarly, we obtain

$$\frac{\Delta z_{t-2}(m)}{w_{t-1}(m)} - f_{t-1}(m)u^{p_2}(m) = \sum_{m_t=0}^{m-1} w_t(m_t)f_t(m_t)u^{p_2}(m_t) = z_{t-1}(m). \quad (2.35)$$

Thus, finally making use of $u^{p_1}(m) \leq z(m)$, we achieve the estimate

$$\Delta z_{t-1}(m) = w_t(m)f_t(m)u^{p_2}(m) \leq w_t(m)f_t(m)z^{\frac{p_2}{p_1}}(m) \leq w_t(m)f_t(m)\Big(\lambda_1 z(m) + \lambda_2\Big). \quad (2.36)$$

If $g(m) \geq 0$ with $g(0) = 0$ for $m \in N_0$, then by applying the summation by parts formula, we have

$$\sum_{n=0}^{m-1} \frac{\Delta g(n)}{\lambda_1 z(n) + \lambda_2} = \frac{g(m)}{\lambda_1 z(m) + \lambda_2} + \sum_{n=0}^{m-1} \frac{\lambda_1 \Delta z(n)}{(\lambda_1 z(n) + \lambda_2)(\lambda_1 z(n+1) + \lambda_2)} \geq \frac{g(m)}{\lambda_1 z(m) + \lambda_2}. \tag{2.37}$$

Using (2.36) and (2.37), we get

$$\frac{z_{t-1}(m)}{\lambda_1 z(m) + \lambda_2} \leq \sum_{m_t=0}^{m-1} \frac{\Delta z_{t-1}(m_t)}{\lambda_1 z(m_t) + \lambda_2} \leq \sum_{m_t=0}^{m-1} w_t(m_t) f_t(m_t). \tag{2.38}$$

Further, using (2.35)-(2.38), we obtain

$$\frac{z_{t-2}(m)}{\lambda_1 z(m) + \lambda_2} \leq \sum_{m_{t-1}=0}^{m-1} w_{t-1}(m_{t-1}) f_{t-1}(m_{t-1}) + \sum_{m_{t-1}=0}^{m-1} w_{t-1}(m_{t-1}) \sum_{m_t=0}^{m_{t-1}-1} w_t(m_t) f_t(m_t). \tag{2.39}$$

Proceeding similarly, we get

$$\frac{z_1(m)}{\lambda_1 z(m) + \lambda_2} \leq \sum_{k=2}^{t} \left(\sum_{m_2=0}^{m-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} w_k(m_k) f_k(m_k) \right). \tag{2.40}$$

Thus

$$\frac{\Delta z(m)}{\lambda_1 z(m) + \lambda_2} \leq w_1(m) f_1(m) + w_1(m) \times \sum_{k=2}^{t} \left(\sum_{m_2=0}^{m-1} w_2(m_2) \cdots \sum_{m_k=0}^{m_{k-1}-1} w_k(m_k) f_k(m_k) \right). \tag{2.41}$$

If we let

$$F(m) = w_1(m) f_1(m) + w_1(m) \times \sum_{k=2}^{t} \left(\sum_{m_2=0}^{m-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} w_k(m_k) f_k(m_k) \right),$$

then (2.41) gives

$$\Delta z(m) \leq \lambda_1 F(m) z(m) + \lambda_2 F(m). \tag{2.42}$$

Now, applying Pachpatte's inequality to (2.42) yields the bound as

$$z(m) \leq z(0) \prod_{m_1=0}^{m-1} (1 + \lambda_1 F(m_1)) + \sum_{m_1=0}^{m-1} \lambda_2 F(m_1) \prod_{n=m_1+1}^{m-1} (1 + \lambda_1 F(n)).$$

Utilizing this bound in $u^{p_1}(m) \leq z(m)$ gives us the desired result.

Remark 2.3. *If $p_1 = 1 = p_2$ is taken into account, the previously described inequality turns into a more extensive nonlinear extension of Pachpatte's inequality (*[17]*, p. 51).*

3. Applications

Example 3.1. In this application, we will investigate the relevance of Theorem 2.3 to determine the boundedness, uniqueness, and continuous dependence of the solutions on the initial data to difference equations of the form

$$L_k u^{p_1}(m) = h(m, u(m)), p_1 \geq 1, \text{ with } u^{p_1}(0) = u_0 \text{ and } L_{t-1} u^{p_1}(0) = 0, \text{for } 2 \leq t \leq k. \tag{3.1}$$

Boundedness of $u(m)$: Assume that $h(m, u(m))$ in equation (3.1) satisfies

$$|h(m, u(m))| \leq f(m)|u(m)|, \tag{3.2}$$

where $f(m)$ is real valued with $f(m) \geq 0, m \in N_0$. Then each solution $u(m)$ of (3.1) is bounded with the explicit bound as

$$|u(m)| \leq \Bigg\{ |u_0| \prod_{m_1=0}^{m-1} \Bigg(1 + \frac{1}{p_1} c^{\frac{1-p_1}{p_1}} \Bigg[w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \sum_{m_3=0}^{m_2-1} w_3(m_3) \ldots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m_k) \Bigg] \Bigg) + \frac{p_1 - 1}{p_1} c^{\frac{1}{p_1}} \sum_{m_1=0}^{m-1} \left(\frac{1}{p_1} c^{\frac{1-p_1}{p_1}} \right)^{m_1} S_{1,m_1+1} \Bigg\}^{\frac{1}{p_1}}, \tag{3.3}$$

where $S_{1,i},\ 1 \leq i \leq m$ are as described in (2.9).

Proof. It is simple to figure out that $u(m)$ satisfies a sum-difference equation,

$$u^{p_1}(m) = u_0 + \sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} h(m_k, u(m_k)), \tag{3.4}$$

provided that $u(m)$ is a solution of (3.1). Using (3.2) in (3.4), we see that

$$|u^{p_1}(m)| \leq |u_0| + \sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m)|u(m)|. \tag{3.5}$$

Then, using Theorem 2.3, we get

$$|u(m)| \leq \Bigg\{ |u_0| \prod_{m_1=0}^{m-1} \Bigg(1 + \frac{1}{p_1} c^{\frac{1-p_1}{p_1}} \Bigg[w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \sum_{m_3=0}^{m_2-1} w_3(m_3) \ldots \sum_{m_{k-1}=0}^{m_{k-2}-1} w_{k-1}(m_{k-1}) \sum_{m_k=0}^{m_{k-1}-1} f(m_k) \Bigg] \Bigg) + \frac{p_1-1}{p_1} c^{\frac{1}{p_1}} \sum_{m_1=0}^{m-1} \left(\frac{1}{p_1} c^{\frac{1-p_1}{p_1}} \right)^{m_1} S_{1,m_1+1} \Bigg\}^{\frac{1}{p_1}}.$$

Uniqueness of $u(m)$: If $h(m, u(m))$ in equation (3.1) satisfies

$$|h(m, u_1(m)) - h(m, u_2(m))| \leq f(m)|u_1^p(m) - u_2^p(m)|, \tag{3.6}$$

then the difference equation (3.1) has at most a single solution.
Proof. If $u_1(m), u_2(m)$ are two solutions of (3.1), using (3.4) and (3.6), we find that

$$|u_1^p(m) - u_2^p(m)| \leq \sum_{m_1=0}^{m-1} w_1(m_1) \sum_{m_2=0}^{m_1-1} w_2(m_2) \cdots \sum_{m_k=0}^{m_{k-1}-1} f(m)|u_1^p(m) - u_2^p(m)|. \tag{3.7}$$

Further, on applying Theorem 2.3 with $u(m) = |u_1^{p_1}(m) - u_2^{p_1}(m)|, u_0 = 0$, and $p_1 = 1 = p_2$ to (3.7), we obtain $u(m) \leq 0$. Thus, consequently, $u(m) = 0$, which gives us that $u_1(m) = u_2(m)$.
Continuous dependency of solution on equation and its initial conditions:
Consider the another difference equation of the class (3.1) as

$$L_k \tilde{u}^{p_1}(m) = \tilde{h}(m, \tilde{u}(m)),\ p_1 \geq 1, \tag{3.8}$$

with $\tilde{u}^{p_1}(0) = \tilde{u}_0$ and $L_{t-1}\tilde{u}^{p_1}(0) = 0$, for $2 \leq t \leq k$. If for small $\epsilon > 0$,

$$|u_0 - \tilde{u}_0| \leq \epsilon \tag{3.9}$$

and

$$P(m)=\sum_{m_1=0}^{m-1}w_1(m_1)\sum_{m_2=0}^{m_1-1}w_2(m_2)\cdots\sum_{m_k=0}^{m_{k-1}-1}f(m_k)|h(m_k,\tilde{u}(m_k))-\tilde{h}(m_k,\tilde{u}(m_k))|\le\epsilon, \tag{3.10}$$

then solution of (3.1) possesses continuous dependency on h and its initial conditions.

Proof. It is clear that equivalent difference equation of (3.8) is

$$\tilde{u}^{p_1}(m)=\tilde{u}_0+\sum_{m_1=0}^{m-1}w_1(m_1)\sum_{m_2=0}^{m_1-1}w_2(m_2)\cdots\sum_{m_{k-1}=0}^{m_{k-2}-1}w_{k-1}(m_{k-1})\sum_{m_k=0}^{m_{k-1}-1}f(m_k)\tilde{u}(m_k),\ m\in N_0. \tag{3.11}$$

For simplicity, let us rewrite right hand sides of equations (3.4) and (3.11) as $u_0+A(m,k,h(m_k,u(m_k)))$ and $\tilde{u}_0+A(m,k,\tilde{h}(m_k,\tilde{u}(m_k)))$, respectively. Then (3.4), (3.6), (3.9), (3.10), and (3.11) altogether gives us that

$$\begin{aligned}|u^{p_1}(m)-\tilde{u}^{p_1}(m)|&\le|u_0-\tilde{u}_0|+A(m,k,|h(m_k,u(m_k))-\tilde{h}(m_k,\tilde{u}(m_k))|)\\&\le|u_0-\tilde{u}_0|+A(m,k,|h(m_k,u(m_k))-h(m_k,\tilde{u}(m_k))\\&\quad+h(m_k,\tilde{u}(m_k))-\tilde{h}(m_k,\tilde{u}(m_k))|)\\&\le|u_0-\tilde{u}_0|+A(m,k,|h(m_k,u(m_k))-h(m_k,\tilde{u}(m_k))|)+P(m)\\&\le 2\epsilon+A(m,k,|u^{p_1}(m_k)-\tilde{u}^{p_1}(m_k)|).\end{aligned} \tag{3.12}$$

Now, Theorem 2.3 with $u(m)=|u^{p_1}(m)-\tilde{u}^{p_1}(m)|$ applied to this inequality produces

$$\begin{aligned}|u^{p_1}(m)-\tilde{u}^{p_1}(m)|\le\Bigg\{&2\epsilon\prod_{m_1=0}^{m-1}\left(1+\frac{1}{p_1}c^{\frac{1-p_1}{p_1}}\left[w_1(m_1)\sum_{m_2=0}^{m_1-1}w_2(m_2)\cdots\sum_{m_k=0}^{m_{k-1}-1}f(m_k)\right]\right)\\&+\frac{p_1-1}{p_1}c^{\frac{1}{p_1}}\sum_{m_1=0}^{m-1}\left(\frac{1}{p_1}c^{\frac{1-p_1}{p_1}}\right)^{m_1}S_{1,m_1+1}\Bigg\}^{\frac{1}{p_1}}.\end{aligned} \tag{3.13}$$

Further, if each of $w_1(m),w_2(m),\ldots,w_k(m)$ are bounded on the subset $\tilde{N}_0$ of N_0, then

$$\begin{aligned}&\prod_{m_1=0}^{m-1}\left(1+\frac{1}{p_1}c^{\frac{1-p_1}{p_1}}\left[w_1(m_1)\sum_{m_2=0}^{m_1-1}w_2(m_2)\cdots\sum_{m_k=0}^{m_{k-1}-1}f(m_k)\right]\right)\\&+\frac{p_1-1}{p_1}c^{\frac{1}{p_1}}\sum_{m_1=0}^{m-1}\left(\frac{1}{p_1}c^{\frac{1-p_1}{p_1}}\right)^{m_1}S_{1,m_1+1}\le M.\end{aligned} \tag{3.14}$$

On combining (3.14) with (3.13), we find that

$$u(m) = |u^{p_1}(m) - \tilde{u}^{p_1}(m)| \leq (2\epsilon M)^{\frac{1}{p_1}}.$$

This indicates the dependence of the solution of (3.1) on ϵ, M, and subsequently on the function f and the initial conditions.

We further offer an example to demonstrate the significance of Theorem 2.7 in determining an explicit bound on its solution.

Example 3.2. Consider

$$\begin{aligned} u^2(m) =& 1 + mu^2(m) + \sum_{m_1=0}^{m-1}\left(\frac{2m_1}{5m_1+1}u^2(m_1)\right) + \left(\sum_{m_1=0}^{m-1}\left(\frac{1}{5m_1+1}\right)\right. \\ & \left.\sum_{m_2=0}^{m_1-1}\left(\frac{2}{1+m_2}\right)\sum_{m_3=0}^{m_2-1}5(m_3+1)u^2(m_3)\right), \end{aligned} \tag{3.15}$$

where $m \in N_0$. Let $u_0 = 1, f_1(m) = 5m+1, f_2(m) = 1+m = f_3(m), w_1(m) = \dfrac{5m-1}{25m^2-1}, w_2(m) = \dfrac{2}{1+m}, w_3(m) = 5$. For $p_1 = 2 = p_2$, an application of Theorem 2.7 gives that

$$u(m) \leq \left\{\prod_{m_1=0}^{m-1}(1+F(m_1))\right\}^{\frac{1}{2}}, \text{ as } \lambda_1 = 1, \lambda_2 = 0, \text{ where} \tag{3.16}$$

$$\begin{aligned} F(m) =& w_1(m)f_1(m) + w_1(m) \\ & \times\left(\sum_{m_2=0}^{m-1}w_2(m_2)f_2(m_2) + \sum_{m_2=0}^{m_1-1}w_2(m_2)\sum_{m_3=0}^{m_2-1}w_3(m_3)f_3(m_3)\right). \end{aligned}$$

Upon computation, using Mathematica, we find that $F(m) = \dfrac{5m^2+9m+2}{10m+2}$. Thus, using the value of $F(m)$ in (3.16), we conclude that

$$\begin{aligned} u(m) \leq& \left\{\prod_{m_1=0}^{m-1}\left(1+\frac{5m_1^2+9m_1+2}{10m_1+2}\right)\right\}^{\frac{1}{2}} \\ =& \left\{\frac{2^{2-m}\Gamma\left(\frac{6}{5}\right)\Gamma\left(m-\frac{\sqrt{281}}{10}+\frac{19}{10}\right)\Gamma\left(m+\frac{\sqrt{281}}{10}+\frac{19}{10}\right)}{\Gamma\left(\frac{1}{10}\left(29-\sqrt{281}\right)\right)\Gamma\left(\frac{1}{10}\left(\sqrt{281}+29\right)\right)\Gamma\left(m+\frac{1}{5}\right)}\right\}^{\frac{1}{2}}. \end{aligned}$$

This provides the bound on the solution $u(m)$ of (3.15) for each $m \in N_0$.

4. Conclusion

This article presents new algebraic discrete inequalities for higher-order finite difference equations with iterated sums using a symmetric function technique. These findings can be used to estimate bounds on the solutions of more broad families of nonlinear finite difference equations with iterated sums, for instances in which the previous results are not directly applicable. A theoretical example of such a difference equation is presented to describe the need to use the symmetric function approach to find its solution and investigate its boundedness, uniqueness, and continuous dependency on initial data. Furthermore, a numerical example is provided to demonstrate the relevance of our results in execution. In the numerical example, the computing program Mathematica is used to determine the explicit bounds of an unknown function. However, for analyzing general classes of difference equations, these results can be further extended to solve and analyze a wider class of higher-order difference equations.

Acknowledgement

The authors would like to express sincere gratitude to the reviewers for their valuable suggestions and comments.

References

[1] Abdeldaim, A. and Yakout, M., On some new integral inequalities of Gronwall-Bellman-Pachpatte type, Applied Mathematics and Computation, 217(6) (2011), 7887-7899.

[2] Akinyele, O., On the Gronwall-Bellman-Bihari type inequalities in several variables with retardation, Journal of Mathematical Analysis and Applications, 104 (1984), 1-26.

[3] Agarwal, R. P. and Thandapani, E., On discrete generalizations of Gronwall's inequality, Bulletin of the Institute of Mathematics Academia Sinica, 9 (1981).

[4] Agarwal, R. P. and Thandapani, E., Some inequalities of Gronwall type, An. St. Univ. Iasi, 28 (1982), 71-75.

[5] Dannan, F., Integral inequalities of Gronwall-Bellman-Bihari type and asymptotic behaviour of certain second order nonlinear differential equations, Journal of Mathematical Analysis and Applications, 108 (1985), 151-164.

[6] Dannan, F., Submultiplicative and subadditive functions and integral inequalities of Bellman-Bihari type, Journal of Mathematical Analysis and Applications, 120 (1986), 631-646.

[7] Gronwall, T. H., Note on the Derivatives with respect to a Parameter of the Solutions of a System of Differential Equations, Annals Of Mathematics, 20 (1919), 292-296.

[8] Kendre, S. D. and Kale, N. R., Some nonlinear finite difference inequalities and its applications, Malaya Journal of Matematik, S (2019), 644-651.

[9] Kendre, S. D. and Kale, N. R., Some generalizations of linear finite difference inequalities of Pachpatte type, Asian-European Journal of Mathematics, 14 (3) (2021).

[10] Li, Z., Some Generalized Gronwall–Bellman Type Difference Inequalities and Applications, Journal of Mathematical Inequalities, 15(3) (2021), 173-200.

[11] Mckee, S., Generalized discrete Gronwall lemmas, Journal of Applied Mathematics and Mechanics, 62 (1982), 429-434.

[12] Medved, M., Bihari Type Inequalities with Multiple Integral and Delay, Periodica Mathematica Hungarica, 7 (1993), 207-212.

[13] Ou-Yang, L., The Boundedness of Solutions of Linear Differential Equations $y'' + A(t)y' = 0$, Advances In Mathematics, 3 (1957), 409-415.

[14] Pachpatte, B. G., On some integral inequalities and their applications to higher order differential equations, Soochow Journal of Mathematics, 14 (1988), 221-230.

[15] Pachpatte, B. G., On a Class of new discrete inequalities, Tamkang Journal of Mathematics, 20 (1991), 19-28.

[16] Pachpatte, B. G., On some inequalities useful in the theory of certain higher order differential and difference equations, Studia University Babes-Bolyai, Mathematica, 40 (1995), 15-34.

[17] Pachpatte, B. G., Inequalities for Finite Difference Equations, Marcel Dekker, New York, 2002.

[18] Qin, Y., Integral and Discrete Inequalities and Their Applications: Volume I: Linear inequalities, Birkhäuser, Springer International Publishing, Switzerland, 2016.

[19] Tsamatos, P. and Ntouyas, S., On a Bellman-Bihari type integral inequality with delay, Periodica Mathematica Hungarica, 23 (1991), 91-94.

[20] Yang, E., On some new discrete inequalities of the Bellman-Bihari type, Nonlinear Analysis, 7 (1983), 1237-1246.

[21] Zhao, C., Some integral inequalities for differential equations, Journal of Binzhou Teachers College, 17 (2001), 41-46.

[22] Zhou, H., Huang, D., Wang, W. and Xu, J., Some new difference inequalities and an application to discrete-time control systems, Journal of Applied Mathematics, (2012).

This page intentionally left blank.